EXAMINATION OF THE VALIDITY OF AN APPROXIMATE SOLUTION OF A CERTAIN VELOCITY EQUATION.

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§1. This problem has been suggested by the important part played by "initial conditions" in the solution of differential equations occurring in Dynamics. In certain cases the special character of the "initial conditions" makes a simple treatment of the equation possible; but sufficient attention has not been paid to the question whether the simplification introduced is legitimate, or to the investigation of the limits within which it holds.

To make the matter clearer, consider the equation—

$$\frac{dy}{dt} = -y + ky^2,$$

with initial condition $y = \lambda$ when $t = 0$, λ being supposed small.

One method of obtaining an approximate solution is to start with—

$$\frac{dy}{dt} = -y$$

and find the solution of this equation. The result is $\lambda\epsilon^{-t}$.

It is then assumed that if the solution thus obtained is small for all values of t, the result given is an approximate solution of the original equation.

The actual solution of the equation is $\lambda \cdot \epsilon^{-t} \cdot \dfrac{1}{1 - \lambda k(1 - \epsilon^{-t})}$, and in order that this be expansible in powers of λ for all values of t it is necessary that $\lambda k < 1$. If $\lambda k > 1$ the solution of the simplified equation is not even approximately a solution of the original equation; and unless λk is a small fraction the solution given will not be a good approximation. The validity of the simplification depends on the nature of the coefficients as well as on the smallness of λ; and it seems important to investigate how small λ must be for the simplified form to be used.

§2. The first question we consider is—

Under what conditions can a solution of—

$$\frac{dy}{dt}=a_1y+a_2y^2 \quad \text{......(1)}$$

with initial condition $y=\lambda$ when $t=0$, be obtained in a series of ascending powers of λ?

[This simple form is taken in order to illustrate the method used in the more general case, and also because the result can be tested by comparison with the solution in finite terms.]

To determine the possibility assume a solution of the form—

$$y=T_0+\lambda T_1+\lambda^2T_2+ \quad \text{......(2)}$$

and determine the T's so as to satisfy the differential equation. If the series thus obtained is convergent the existence of a solution of this form is established.

From the initial conditions we have—

$$T_0=0,\ T_1=1,\ T_2=0,\ T_3=0,\ \dots \text{ when } t=0.$$

Substitute from (2) in (1)—

$$\therefore\ T_0'+\lambda T_1'+\lambda^2T_2'+\dots=a_1(T_0+\lambda T_1+\lambda^2T_2+\dots)$$
$$+a_2(T_0+\lambda T_1+\lambda^2T_2\dots)^2,$$

where T_r' means $\frac{dT^r}{dt}$.

Equating powers of λ—

$$\left.\begin{aligned} T_0'&=a_1T_0+a_2T_0^2\\ T_1'&=a_1T_1+a_2\,.\,2T_0T_1\\ T_2'&=a_1T_2+a_2(T_1^2+T_0\,.\,T_2)\\ &\&c. \end{aligned}\right\} \quad \text{......(3)}$$

The first of these has the initial condition $T_0=0$ when $t=0$; hence initially $T_0'=0$; and thus $T_0=0$ for all time.

Transform the set (3) by the substitution—

$$T_r=S_r\,.\,\varepsilon^{a_1t} \quad \text{......(4)}$$

and the equations (3) become a set of which a typical member is—

$$S_r'=a_2\,.\,\varepsilon^{a_1t}[S_0S_n+S_1S_{r-1}+\dots+S_{r-1}\,.\,S_1+S_rS_0] \quad \text{......(5)}$$

The initial condition applicable to T_1 gives us $S_1=1$ when $t=0$.

Further substitute—

$$S_r=a_2^{r-1}R_r \quad \text{......(6)}$$

for values of r above unity and the typical number (4) becomes—

$$R'_r = \varepsilon^{a_1 t}[R_0 R_r + R_1 R_{2-1} \dots + R_{2-1} R_0] \quad \dots\dots\dots\dots\dots\dots (7)$$

Hence, using the initial conditions—

$$R_r = \int_0^t \varepsilon^{a_1 t}[R_0 R_r + R_1 R_{r-1} \dots + R_{r-1} R_1 + R_r R_0]dt \quad \dots\dots\dots (8)$$

It is here to be noted that $R_0 = 0$; and hence when the expansion (2) is applicable the values of the coefficients can be determined by quadratures from (8).

The solution is—

$$\lambda\varepsilon^{a_1 t}[1 + a_2\lambda R_2 + a_2^2\lambda^2 R_3 + \dots] \dots\dots\dots\dots\dots\dots (9)$$

To discuss the convergence of (9).

From the values of R_0 and R_1, and the mode of formation of the subsequent coefficients, it is clear that the R's are all positive for positive values of t.

$\varepsilon^{a_1 t}$ is one factor in the solution, and will occur to higher powers in the other factor; so that, if we include in our consideration large values of t, a necessary condition for the convergence of the series is a_1 negative.

Put $a_1 = -\alpha_1$ where α_1 is positive.

Applying this result we have—

$$\left.\begin{aligned}
R_0 &= 0 \\
R_1 &= 1 \\
R_2 &= \int_0^t \varepsilon^{a_1 t} . R_1 . R_1 < \frac{1}{\alpha_1} \\
R_3 &= \int_0^t \varepsilon^{a_1 t}(R_1 R_2 + R_2 R_1) < \frac{1}{\alpha_1}(K_2 K_1 + K_1 K_2) \\
&\vdots \\
R_n &< \frac{1}{\alpha}\{K_{n-1}K_1 + K_{n-2}K_2 \dots + K_2 . K_{n-2} + K_1 K_{n-1}\}
\end{aligned}\right\} \dots\dots (10)$$

where the K's satisfy the relations—

$$\left.\begin{aligned}
K_0 &= 0 \\
K_1 &= 1 \\
K_2 &= \frac{1}{\alpha_1} \\
K_3 &= \frac{1}{\alpha_1}(K_2 K_1 + K_1 K_2) \\
K_r &= \frac{1}{\alpha_1}(K_{r-1}K_1 + K_{r-2}K_2 \dots + K_2 K_{r-2} + K_1 K_{r-1})
\end{aligned}\right\} \dots\dots\dots (11)$$

This can be simplified still further by putting—

$$K_r = \frac{L_r}{a_1^{r-1}} \text{ (excluding } r=0.$$

We now have—

$$\left.\begin{aligned} &L_0 = 0 \\ &L_1 = 1 \\ &L_2 = 1 \\ &L_3 = (L_2L_1 + L_1L_2) \\ &L_n = (L_{n-1}L_1 + L_{n-2}L_2 \ldots + L_2L_{n-2} + L_1L_{n-1}) \end{aligned}\right\}\cdots\cdots\cdots\cdots$$

To determine the L's put—

$$z = L_2 + L_3x + L_4x^2 + \ldots$$

so that $1 + xz = L + L_2x + L_3x^2 + \ldots$

$$\begin{aligned} x^2(1+x^2)^2 &= (L_0 + L_1x + L_2x^2 \ldots)^2 \\ &= L_0^2 + x(L_0L_1 + L_1L_0) + x^2(L_0L_2 + L_1L_1 + L_2L_0) + \ldots \\ &= L_2x^2 + L_3x^3 + L_3x^4 \\ &= x^2z \\ &\therefore\ (1+x^2)^2 = {}^2 \\ &\therefore\ z^2x^2 + {}^2(2x-1) + 1 = 0 \\ &\therefore\ z = \frac{1 - x2 - \sqrt{1-4x}}{2x^2}, \text{ selecting the appropriate sign} \\ &= \frac{1}{2}c_2 + \frac{1}{2}c_3 \cdot x + \frac{1}{2} \cdot c_4x^2 + \ldots \end{aligned}$$

where c_r is the coefficient of $-x^r$ in the expansion of $(1-4x)^{\frac{1}{2}}$.

Hence $L_r = \frac{1}{2}c_r$.

Now the expansion of $(1-4x)^{\frac{1}{2}}$ in ascending powers of x is convergent if $x < \frac{1}{4}$; hence $\underset{n=\infty}{L} \frac{c_{n+1}}{c_n} \ngtr 4$, and $\therefore \underset{n=\infty}{L} \frac{L_{n+1}}{L_n} \ngtr 4$.

Thus the solution of the differential equation is—

$$\lambda\varepsilon - a_1t\left[1 + \frac{a_2\lambda}{a_1} \cdot H_2 + \frac{a_2^2\lambda^2}{a_1^2}H_3 \ldots\right]$$

where $L\frac{H_{n+1}}{H_n} \ngtr 4$.

This will therefore be convergent if $\left|\frac{a_2\lambda}{a_1}\right| < \frac{1}{4}$.

Hence conditions which are certainly sufficient for the applicability of this form are—

(1) a_1 negative.

(2) $\lambda < \left| \frac{a_1}{4a_2} \right|$.

(For the case considered in §1 this gives $\lambda < \frac{1}{4k}$, whereas we know that the range of availability of the method is bounded by $\lambda < \frac{1}{k}$).

§3. Apply the same method to the more general equation—

$$\frac{dy}{dt} = a_1 y + a_2 y^2 + a_3 y^3 + \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

subject to the initial condition $y = \lambda$ when $t = 0$.

Try a solution of the form—

$$y = T_0 + \lambda T_1 + \lambda^2 T_2 + \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Substitute in (13) and equate coefficients of λ—

$$\left.\begin{aligned} \therefore T'_0 &= a_1 T_0 + a_2 T_0^2 + a_3 T_0^3 + \ldots \\ T'_1 &= a_1 T_1 + a_2(T_0 T_1 + T_1 T_0) + a_3 \,.\, 3T_0^2 T_1 + \ldots \\ T'_1 &= a_1 T_2 + a_2(T_0 T_2 + T_1^2) + a_3 \,.\, 3T_1^2 \,.\, T_0 + \ldots \\ &\&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots(15)$$

The initial conditions for the different T's are—

$$T_0 = 0, \ T_1 = 1, \ T_2 = 0, \ T_3 = 0 \ \ldots \text{ when } t = 0.$$

Substitute $T_r = \varepsilon^{a_1 t} S_r$—

$$\left.\begin{aligned} \therefore S_0 &= 0 \\ S_1 &= 1 \\ S'_2 &\varepsilon^{a_1 t} \,.\, a_2(S_0 S_2 + S_1^2) + \varepsilon^{2\sigma_1 t} \,.\, a_3(3S_0 S_1^2) + \ldots \\ S'_3 &= \varepsilon^{a_1 t} \,.\, a_2(2S_1 S_2) + \varepsilon^{2a_1 t}(3S_0 S_3 + 6S_1 S_2 S_0) + \ldots \end{aligned}\right\} \ldots\ldots\ldots(16)$$

Once more it is to be noticed that the coefficients T or S can be immediately evaluated by quadratures, since the coefficient of T_r in the equation for T'_r is zero, and the earlier T's are calculated in succession.

As in the case discussed in §2, a first condition for convergence will be a_1 negative, equal to $-\alpha_1$ (say) where α_1 is positive. Follow the line taken in §2, putting $|a_2| = \alpha_2$, $|a_3| = \alpha_3$, &c.; we then find that each of the S's has, for all values of T, a modulus not greater than that of the corresponding R where the set of R's is given by the equations—

$$\left.\begin{aligned} R_0 &= 0 \\ R_1 &= 1 \\ R_2 &= \frac{a_2}{a_1}(R_0R_2 + R_1^2) + \frac{a_3}{2a_1}(3R_0R_1^2) + \ldots \\ R_3 &= \frac{a_2}{a_1} \cdot 2R_1R_2 + \frac{a_3}{2a_1}(3R_0R_3 + 6R_1R_2R_0) + \ldots \\ &\&c \end{aligned}\right\} \ldots\ldots\ldots\ldots(17)$$

Write—

$$\eta = R_1\xi + R_2\xi^2 + R_3\xi^3 + \ldots$$

Then $\frac{a_2}{a_1} \cdot \eta^2 + \frac{a_3}{2a_1} \cdot \eta^3 + \frac{a_4}{3a_1} \cdot \eta^4 + \ldots$

$$\begin{aligned} &= \frac{a_2}{a_1}(R_1\xi + R_2\xi^1 + \ldots)^2 + \frac{a_3}{2a_1}(R_1\xi + R_2\xi^2 + \ldots)^3 + \ldots \\ &= R_2\xi^2 + R_3\xi^3 + R_4\xi^4 + \ldots \\ &= \eta - R_1\xi \\ &= \eta - \xi \end{aligned}$$

$$\therefore \xi = \eta - \frac{a_2}{a_1} \cdot \eta^2 - \frac{a_3}{2a_1} \cdot \eta^3 - \frac{a_4}{3a_1} \cdot \eta^4 \ldots$$

The reversion of this series gives an expression for η in ascending powers of ξ, the coefficients therefore being the R's.

In a previous paper * I have shown that the reversed series is convergent if $|\xi| < J$, where $J = \sqrt{(1+2\beta)^2+1} \quad -(1+2\beta)$, β being the greatest of the set $\frac{a_2}{a_1}, \frac{a_3}{2a_1}, \frac{a_4}{3a_1} \ldots$

Hence finally the solution of—

$$\frac{dy}{dt} = a_1y + a_2y^2 + a_3y^3 + \ldots$$

with initial condition $y = \lambda$ when $t = 0$ can certainly be obtained by successive approximation in a convergent series of powers of λ provided

a_1 is negative,

and $|\lambda| < \sqrt{1+2\beta)^2+1} \quad -(1+2\beta)$,

where β is the greatest of the set $\left|\frac{a_2}{a_1}\right|, \left|\frac{a_3}{2a_1}\right|, \left|\frac{a_4}{3a_1}\right| \ldots$

* Brit. Assoc. Report, 1905, p. 319.